Stabilizing unstable discrete systems

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A general method for stabilizing unstable discrete systems to a fixed point or high-period orbit is developed analytically and numerically in this paper. It is shown that the method can be equally applied to the systems with one or more positive Lyapunov exponents. Moreover, the method does not require a prior analytical knowledge of the system under investigation, nor any additional control parameters.

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I. INTRODUCTION

In recent years, the research of stabilizing unstable system to a fixed point or high-period orbit, i.e., controlling chaos, has attracted much attentions in nonlinear sciences and particularly in physics, chemistry, and biology. A large number of control methods have been developed and are being applied to real systems [1–11]. Many of them are extensions or generalizations of the original work of Ott, Grebogi, and York (OGY) [1]. The OGY method and its variants are based on a parametric perturbation mechanism. And at least one accessible tuning parameter is required in advance for using the OGY method or its variants. However, in many practical situations, such a parameter often cannot be found at all. In addition, most of the methods are designed for (or restricted to) such unstable systems that have just one positive Lyapunov exponent. Recently, Yang, Liu, and Mao (YLM) [12] presented one new method. The YLM method can be successfully applied to control the unstable systems with multiple positive Lyapunov exponents, i.e., the so-called hyperchaos. On the other hand, the YLM method still makes use of a parametric perturbation as like the OGY method.

In this paper, we pursue to develop a method that can be applied to both chaos and hyperchaos. Furthermore, the proposed method does not require any adjustable control parameters of the system.

The paper is organized as follows. In Sec. II, the mechanism of the control method is analyzed in detail. In Sec. III, we extend the method introduced in Sec. II to the stabilization of higher-period orbit. Then several typical chaotic and hyperchaotic systems are taken as numerical examples to illustrate the applicability of the proposed method in Sec. IV. At the end of the paper, some discussions and conclusions are given.

II. THE METHOD

Consider an n-dimensional dynamical system defined by

$$\mathbf{x}_{k+1}=\mathbf{F}(\mathbf{x}_k), \tag{1}$$

where $\mathbf{x}\in\mathbf{R}^n$ is an n-dimensional vector, $\mathbf{F}$ is a nonlinear vector valued function.

Let $\mathbf{x}_f$ be the fixed point of the map (1). In order to stabilize a chaotic orbit to this fixed point, we take a variable feedback control described by

$$\mathbf{x}_{k+1}=\mathbf{F}(\mathbf{x}_k)+\mathbf{M}(\mathbf{F}(\mathbf{x}_k)-\mathbf{x}_k), \tag{2}$$

where $\mathbf{M}$ is an $n\times n$ matrix that we are going to determine definitely in this paper. Equation (2) takes the form of the so-called adaptive adjustment mechanism (AAM) introduced in Ref. [13]. However, the matrix $\mathbf{M}$ is restricted to be a diagonal matrix there. This is not the case in our method. In terms of the eigenvalue analysis [13], it is found that AAM can be applied to some special types of the fixed points, and even by using the so-called nonuniformly AAM. In this work, we take a similar idea as Yang, Liu, and Mao [12] to determine the matrix $\mathbf{M}$. Now let us define an infinitesimal deviation of $\mathbf{x}_k$ from $\mathbf{x}_f$ as $\delta\mathbf{x}_k=\mathbf{x}_k-\mathbf{x}_f$. Then from Eq. (2), one has

$$\delta\mathbf{x}_{k+1}\approx\mathbf{J}\,\delta\mathbf{x}_k+\mathbf{M}(\mathbf{J}-\mathbf{I})\,\delta\mathbf{x}_k, \tag{3}$$

where $\mathbf{J}=(\partial\mathbf{F}/\partial\mathbf{x}_k)|_{\mathbf{x}_k=\mathbf{x}_f}$ is the Jacobian matrix of the original system $\mathbf{F}$ evaluated at the fixed point $\mathbf{x}_f$ and $\mathbf{I}$ is the $n\times n$ identity matrix. In practice, the matrix $\mathbf{J}$ is experimentally accessible by taking the well-known embedding technique [1,14]. The goal of controlling here is to make $\lim_{k\to\infty}|\delta\mathbf{x}_k|\to 0$ (which implies that $\mathbf{x}_k\to\mathbf{x}_f$, as $k\to\infty$). For this aim, we require

$$\delta\mathbf{x}_{k+1}=\mathbf{Q}\,\delta\mathbf{x}_k, \tag{4}$$

where $\mathbf{Q}$ is an $n\times n$ matrix and takes the form

$$\mathbf{Q}=\begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix}, \tag{5}$$

where $q_1,q_2\in(-1,1)$ are constants. Substituting Eq. (4) into Eq. (3) and eliminating $\delta\mathbf{x}_k$, we have

$$\mathbf{M}=(\mathbf{Q}-\mathbf{J})(\mathbf{J}-\mathbf{I})^{-1}, \tag{6}$$

where we have assumed that the inverse matrix $(\mathbf{J}-\mathbf{I})^{-1}$ exists. One special form of the matrix $\mathbf{Q}$ is $\mathbf{Q}=q\mathbf{I}$, i.e., by setting $q_1=q_2=q$. Then the matrix $\mathbf{M}$ becomes

$$\mathbf{M}=(q\mathbf{I}-\mathbf{J})(\mathbf{J}-\mathbf{I})^{-1}, \tag{7}$$

where q is a constant and $q\in(-1,1)$ as mentioned above.

Note that although it is just the discrete time systems that are discussed by now, the present approach can also be applied to control the flows just by taking the corresponding Poincaré sections.

The control method as given by Eq. (7) does not require any prior analytical knowledge of the system under investigation, since the elements of matrix **J** can be obtained from experimental data by using the known embedding technique. As concerns the size of converging region, it is also under investigation and will be reported elsewhere. In addition, similar to the YLM method, our method is also formulated for an n-dimensional system with n being an integer. So our method can then be applied to any finite-dimensional system in principle, including chaotic and hyperchaotic systems. Furthermore, by choosing an appropriate value of q between -1 and 1, one may have an optimal control through Eq. (7). In addition, it should be noted that the method proposed here cannot be applied to the cases in which one eigenvalue of **J** equals 1 or even some eigenvalues are close to 1 from practical point of view.

On comparing with the YLM method, first, our method does not require any adjustable controlling parameters in advance, and so it can be applied to much more extensive systems. Second, once the constant q is chosen in the range of $(-1, 1)$, then **M** is definitely determined and need not to be changed with the discrete time. Therefore it is much simpler to implement.

On the other hand, by comparison with the AAM, the main progress of our method is that it can be equally applied to different types of the fixed points particularly to those ones to which AAM cannot be applied at all, as illustrated in our examples.

III. HIGH-PERIOD ORBIT

The method developed in Sec. II can also be applied to stabilize a high-period orbit. Assuming one want stabilize a period-p orbit, i.e., the orbit $\{\mathbf{x}^1,\ldots,\mathbf{x}^p\}(p>1)$. By replacing $\mathbf{F}(\mathbf{x}_k)$ with $\mathbf{F}^{(p)}(\mathbf{x}_k)$ in Eq. (2), one gets

$$\mathbf{x}_{k+1}=\mathbf{F}^{(p)}(\mathbf{x}_k)+\mathbf{M}(\mathbf{F}^{(p)}(\mathbf{x}_k)-\mathbf{x}_k), \qquad (8)$$

where $\mathbf{F}^{(p)}(\mathbf{x}_k)$ denotes p times iterations of $\mathbf{F}(\mathbf{x}_k)$. Repeating the process in Sec. II, we finally have equation analogous to Eq. (6) or Eq. (7) as

$$\mathbf{M}=(\mathbf{Q}-\tilde{\mathbf{J}})(\tilde{\mathbf{J}}-\mathbf{I})^{-1} \qquad (9)$$

or

$$\mathbf{M}=(q\mathbf{I}-\tilde{\mathbf{J}})(\tilde{\mathbf{J}}-\mathbf{I})^{-1}, \qquad (10)$$

where

$$\tilde{\mathbf{J}}=\left(\frac{\partial \mathbf{F}^{(p)}(\mathbf{x}_k)}{\partial \mathbf{x}_k}\right)_{\mathbf{x}_k=\mathbf{x}^1}$$

is the Jacobian matrix of $\mathbf{F}^{(p)}(\mathbf{x}_k)$ evaluated at $\mathbf{x}^1$, $\mathbf{Q}$ takes the form of Eq. (5) and $q\in(-1,1)$ is a constant as mentioned previously. It is easy to know that

$$\tilde{\mathbf{J}}=\prod_{i=1}^{p}\mathbf{J}_i, \qquad (11)$$

where

$$\mathbf{J}_i=\left(\frac{\partial \mathbf{F}}{\partial \mathbf{x}_k}\right)_{\mathbf{x_k}=\mathbf{x^i}}, \quad i=1,\ldots,p.$$

The question now is how to locate the unstable period orbit. In practice, there is lot of literature concerning this problem [15–18]. In this work, we take the algorithm presented by Schmelcher and Diakonos [18].

IV. NUMERICAL EXAMPLES

The control method given by Eq. (6) or Eq. (7) can be applied to both chaotic and hyperchaotic systems. In the following, we take two systems as examples. We first discuss a chaotic map and then a hyperchaotic map to illustrate our method.

A. Controlling chaos

Consider the Hénon map [19] described by

$$x_{k+1}=a-x_k^2+by_k,$$
$$y_{k+1}=x_k, \qquad (12)$$

where a,b are the parameters, and we choose $a=1.4$ and $b=0.3$ in this work. This map has two fixed points: $\mathbf{x}_f^{(1)}\approx(0.883\,89,0.883\,89)$ and $\mathbf{x}_f^{(2)}\approx(-1.583\,89,-1.583\,89)$. In Ref. [12], they belong to two different types of fixed points and require to be stabilized separately according to the simple and nonuniformly AAM. This is not the case in this research. Two fixed points can be equally dealt with by using the method proposed above. Here we just take the second one as an application. The Jacobian matrix corresponding the fixed point $\mathbf{x}_f^{(2)}$ is

$$\mathbf{J}=\begin{pmatrix}-2x_f & b\\ 1.0 & 0.0\end{pmatrix},$$

where $x_f=-1.583\,89$ and $b=0.3$. Then from Eq. (7), we have

$$\mathbf{M}=\begin{pmatrix}\dfrac{q+2x_f-b}{-2x_f-1+b} & \dfrac{(q-1)b}{-2x_f-1+b}\\[2ex] \dfrac{q-1}{-2x_f-1+b} & \dfrac{q(2x_f+1)-b}{-2x_f-1+b}\end{pmatrix},$$

where $q\in(-1,1)$ is a constant. Choosing the parameter $q=0.5$, one gets

$$\mathbf{M}=\begin{pmatrix}-1.202\,61 & -0.060\,78\\ -0.202\,61 & -0.560\,78\end{pmatrix}.$$

Then the equation analogous to Eq. (2) is

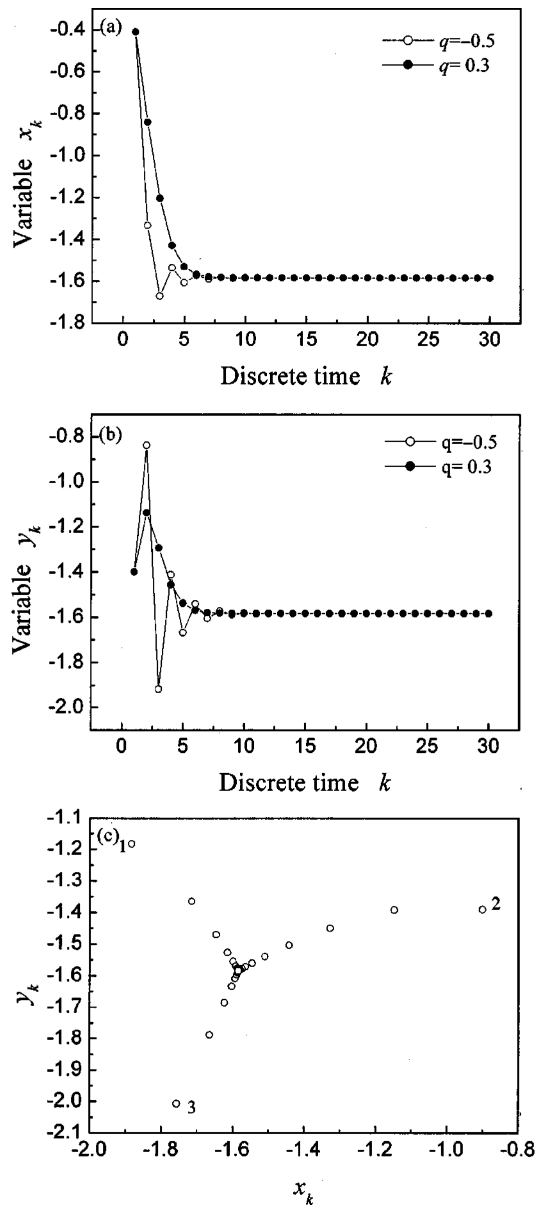

FIG. 1. Numerical results of controlling Hénon map. (a) x_k versus k for $q=-0.5$ and 0.3, respectively; (b) y_k versus k for $q=-0.5$ and 0.3; (c) Three orbits starting from different initial points are stabilized to the fixed point $(-1.583\,89,\ -1.583\,89)$, for $q=0.5$.

$$x_{k+1}=a-x_k^2+by_k-1.202\,61(a-x_k^2+by_k-x_k)-0.060\,78(x_k-y_k),$$

$$y_{k+1}=x_k-0.202\,61(a-x_k^2+by_k-x_k)-0.560\,78(x_k-y_k). \tag{13}$$

Evolving this controlling system from an arbitrary initial point in attracting basin, it is found that the orbit is stabilized to the fixed point $\mathbf{x}_f^{(2)}$ monotonically. The numerical results are shown in Fig. 1. In Figs. 1(a) and 1(b), x_k versus k and y_k versus k are plotted for two different parameters q. In Fig. 1(c), three orbits starting from different initial points are stabilized to the desired fixed point.

B. Controlling hyperchaos

In order to compare our method with the YLM method and the AAM more definitely, let us discuss the following map [20] described by

$$x_{k+1}=1-2(x_k^2+y_k^2)+p,$$

$$y_{k+1}=-4x_ky_k+q. \tag{14}$$

To be an illustrative example, this map can be investigated more analytically. In Ref. [12], the parameters p and q are taken to be the adjustable controlling parameters. And by adjusting p and q, the unstable orbit is stabilized to the desired fixed point. However, we may take $p=q=0$ in this work. Now, there exist four different fixed points for map (14). Here we take only one of them, i.e., the point $(-0.25, 0.75)$, as an application. For this fixed point, the Jacobian matrix is

$$\mathbf{J}=\begin{pmatrix} 1.0 & -3.0 \\ -3.0 & 1.0 \end{pmatrix}.$$

The two eigenvalues of $\mathbf{J}$ are $\lambda_1=-2.0$ and $\lambda_2=4.0$, respectively. According to Eq. (7), one knows that

$$\mathbf{M}=\begin{pmatrix} -1.0 & -\frac{q-1}{3} \\ -\frac{q-1}{3} & -1.0 \end{pmatrix},$$

where q is a constant and $q\in(-1,1)$. The two eigenvalues of $\mathbf{M}$ are $-1+(q-1)/3$ and $-1-(q-1)/3$, respectively. Since the constant satisfies $q\in(-1,1)$, the eigenvalue $-1-(q-1)/3\in(-1,1)$. That is, one of the unstable directions becomes stable under the control. Then the equation analogous to Eq. (2) is

$$x_{k+1}=1-2(x_k^2+y_k^2)-[1-2(x_k^2+y_k^2)-x_k]-\frac{q-1}{3}(-4x_ky_k-y_k),$$

$$y_{k+1}=-4x_ky_k-\frac{q-1}{3}[1-2(x_k^2+y_k2)-x_k]-(-4x_ky_k-y_k). \tag{15}$$

The numerical results are shown in Fig. 2, for $q=0.5$. In Figs. 2(a) and 2(b), the curves of x_k versus k and y_k versus k are plotted, respectively. And in Fig. 2(c), three orbits starting from different initial points are stabilized to the fixed point $(-0.25, 0.75)$. It is shown that the unstable orbit is stabilized to the desired fixed point monotonically.

Now, for an illustration, we show how a period-2 orbit $\{\mathbf{x}^1,\mathbf{x}^2\}$ of system (14) is stabilized with the help of the ap-

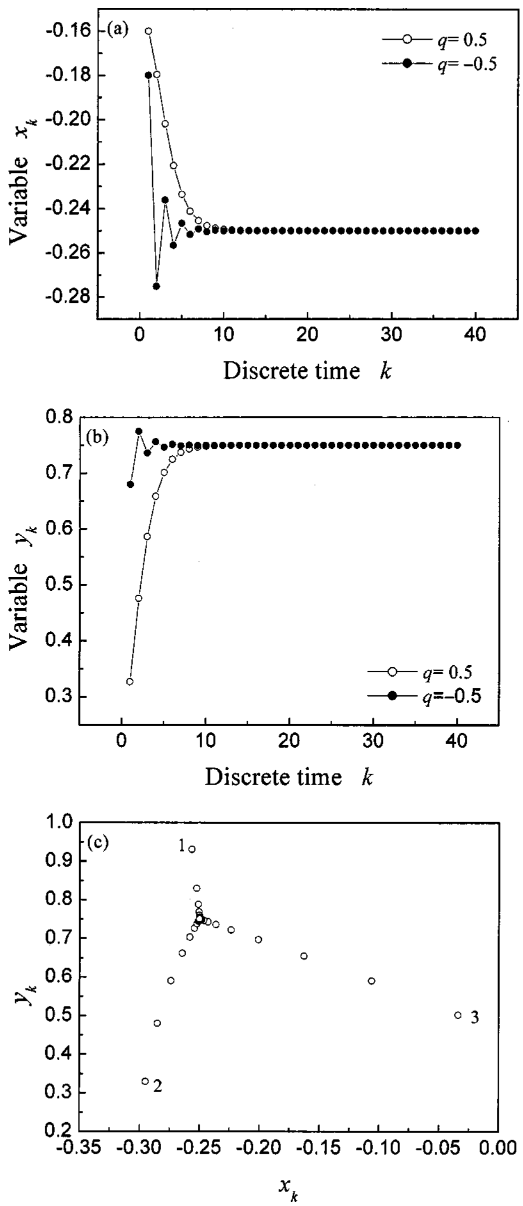

FIG. 2. Numerical results of controlling hyperchaotic system (14). (a) x_k versus k for $q=0.5$ and -0.5; (b) y_k versus k for $q=0.5$ and -0.5; (c) Three orbits starting from different initial points are stabilized to the fixed point $(-0.25, 0.75)$, for $q=0.5$.

proach described in Sec. III, here $\mathbf{x}^1 \approx (0.809\,02, 0.0)$ and $\mathbf{x}^2 \approx (-0.309\,02, 0.0)$. For point $\mathbf{x}^1$, it is easy to get

$$\mathbf{J}_1 = \begin{pmatrix} -3.236\,08 & 0.0 \\ 0.0 & -3.236\,08 \end{pmatrix}$$

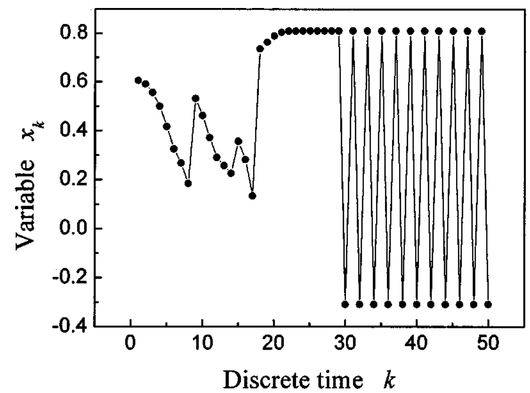

FIG. 3. Stabilizing a unstable period-2 orbit of hyperchaotic system (14). Here we choose $q=0.1$ in the computation. When the difference between two neighboring iterations is smaller than 10^{-10}, the control imposed on the system is removed.

and

$$\mathbf{J}_2 = \begin{pmatrix} 1.236\,08 & 0.0 \\ 0.0 & 1.236\,08 \end{pmatrix}.$$

Then in terms of Eq. (10), one has

$$\mathbf{M} = \begin{pmatrix} -0.2q-0.8 & 0.0 \\ 0.0 & -0.2q-0.8 \end{pmatrix}, \qquad (16)$$

where $-1 < q < 1$ is a constant as mentioned previously. After making use of Eq. (8), it is found that the desired point is successfully stabilized as shown in Fig. 3. In Fig. 3, the control imposed on the system is removed and the system let evolve freely, when the error between the two near iterations is smaller than 10^{-10}.

V. CONCLUSION

In this work, we show how an unstable system, with one or more positive Lyapunov exponents, is stabilized by using a different general method. It is found that the proposed method neither requires a prior analytical knowledge of the underlying system nor any adjustable control parameters in advance. Therefore, it can be applied to a very large range of systems, in particular, hyperchaotic systems.

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